EXPANSION OF GENERALIZED STIELTJES CONSTANTS IN TERMS OF DERIVATIVES OF HURWITZ ZETA-FUNCTIONS

Marc Prévost

EXPANSION OF GENERALIZED STIELTJES CONSTANTS IN TERMS OF DERIVATIVES OF HURWITZ ZETA-FUNCTIONS.

M. PRÉVOST

Abstract. Generalized Stieltjes constants $\gamma_n(a)$ are the coefficients in the Laurent series for the Hurwitz-zeta function $\zeta(s,a)$ at the pole $s=1$. Many authors proved formulas for these constants. In this paper, using a recurrence between $(\zeta(s+j,a))_j$ and proved by the author, we prove a general result which contains some of these formulas as particular cases.

1. Introduction

For $a \in \mathbb{C}$, $\Re(a) > 0$, the Hurwitz zeta function is defined as

$$\zeta(\sigma,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^{\sigma}}, \quad \Re(\sigma) > 1.$$

It can be analytically continued to $\sigma \in \mathbb{C} \setminus \{1\}$, with a pole at $\sigma = 1$. The generalized Stieltjes constants $\gamma_n(a)$ occur as coefficients in the Laurent series expansion of $\zeta(\sigma,a)$ at the pole $\sigma = 1$:

$$\zeta(\sigma,a) = \frac{1}{\sigma-1} + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \gamma_{\ell}(a)(\sigma-1)^{\ell}. \tag{1}$$

For $a=1$, $\zeta(\sigma,a)$ is the usual zeta-function $\zeta(\sigma)$.

There are numerous representations for them, for instance [3, 9],

$$\gamma_{\ell}(a) = \lim_{m\to\infty} \left\{ \sum_{k=0}^{m} \frac{\ln^{\ell}(k+a)}{k+a} - \frac{\ln^{\ell+1}(m+a)}{\ell+1} \right\}, \quad \ell = 0,1,2,\ldots, a \neq 0,-1,-2,\ldots$$

If $a=1$, the generalized Stieltjes constant $\gamma_{\ell}(a)$ is the usual Stieltjes constant γ_{ℓ}.

The series for Hurwitz zeta function converges absolutely for $\Re(\sigma) = \alpha > 1$ and the convergence is uniform in the half plane $\alpha \geq \alpha_0 > 1$. So $\zeta(\sigma,a)$ is analytic in the half-plane $\Re(\sigma) = \alpha > 1$.

Numerical approximations of $\gamma_{\ell}(a)$ have been recently given in [2, 6, 8]. In the last reference, Padé approximation of the remainder term of a series giving Stieltjes constant provides a new approximation of this constant.

Date: May 24, 2023.

2020 *Mathematics Subject Classification.* Primary 11Y60, 41A21; Secondary 41A60.

Key words and phrases. Stieltjes constants, Riemann zeta function, Hurwitz zeta function, Laurent expansion, Stirling numbers of the first kind.

The goal of the current paper is to write new series for $\gamma_\ell(a)$ whose terms are the derivatives of the zeta function at integers. These results will generalize the formula for Euler constant proved in [5].

In [7], using the Padé approximation $[n,m]_{e^x} = \frac{\sum_{j=0}^{n} \frac{(-n)_j}{(-m-n)_j}\frac{x^j}{j!}}{\sum_{j=0}^{m} \frac{(-m)_j}{(-m-n)_j}\frac{(-x)^j}{j!}}$ of the exponential function in the following integral representation of $\zeta(\sigma, a)$,

$$\zeta(\sigma, a) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{x^{\sigma-1}e^{-ax}}{1-e^{-x}} dx \tag{2}$$

we proved the following result:

For $n, m \in \mathbb{N}, \sigma \in \mathbb{C} \setminus D_{n,m}$ where $D_{n,m} := \{-n-m, 1-\max(n,m), \cdots, -1, 0, 1\}$, for $(p, a) \in \mathbb{C}^2$, let us define the quantity $A_{(n,m)}^{(p)}(\sigma, a)$ by

$$A_{(n,m)}^{(p)}(\sigma, a) := \sum_{j=0}^{m} \frac{(-m)_j}{(-n-m)_j} p^j \frac{(\sigma)_j}{j!} \zeta(\sigma+j, a+p) - \sum_{j=0}^{n} \frac{(-n)_j}{(-n-m)_j} (-p)^j \frac{(\sigma)_j}{j!} \zeta(\sigma+j, a). \tag{3}$$

Theorem 1. *If $\Re(a) > 0$ and $\Re(a+p) > 0$, then*

$$A_{(n,m)}^{(p)}(\sigma, a) = (-1)^{n+1} p^{m+n+1} \frac{(\sigma)_{m+n+1}}{(n+m)!} \int_0^1 x^m (1-x)^n \zeta(\sigma+m+n+1, a+px) dx$$

where $(\sigma)_j := \sigma(\sigma+1)\cdots(\sigma+j-1) = \frac{\Gamma(\sigma+j)}{\Gamma(\sigma)}$ is the Pochhammer symbol.

Remark 1. The conditions $\Re(a) > 0$ and$\Re(a+p) > 0$ are not restrictive since $\zeta(\sigma, a) = \zeta(\sigma, a+k) + \sum_{i=0}^{k-1} (i+a)^{-\sigma} > 0$ and $\Re(a+k) > 0$ for suitable integer k.

The convergence of $A_{(n,m)}^{(p)}$ when n or m tends to infinity has been proved by the following Lemma.

Lemma 1. *If $|p| \leq |a|, |p| \leq |a+p|, \Re(a) > 0$ and $\Re(a+p) > 0$, then $\forall \sigma \in \mathbb{C} \setminus \mathbb{Z}^-, \sigma \neq 1$*

$$\left| \int_0^1 x^m (1-x)^n \zeta(\sigma+m+n+1, a+px) dx \right| \leq C \frac{1}{|a+p|^m} \frac{1}{|a|^n}$$

where C is some constant independent of m and n.

So, under conditions on a and p, the previous lemma shows that the quantity $A_{(n,m)}^{(p)}(\sigma, a)$ tends to 0 when m or n tends to infinity.

In this paper, we use Theorem 1 to construct formulas for Stieltjes constants. To do that, we will use the following expression for Stieltjes constant

$$\gamma_\ell(a) = (-1)^l \left(\frac{d}{d\sigma}\right)^\ell \left(\zeta(\sigma, a) - \frac{1}{\sigma-1}\right)\Bigg|_{\sigma=1}.$$

2. Results

In this section, we prove a general formula for Stieltjes constant which depends on two parameters: n is the degree of the numerator and m the degree of the denominator of the Padé approximant used to approximate the function e^{-x} in (2).

We denote by $\zeta^{(r,0)}(j,a)$ the r-th derivative of the function $\zeta(\sigma,a)$ with respect to the variable σ, computed at $\sigma = j$.

Theorem 2. *Suppose that $\Re(a) > 0$ and $|a| \geq 1$. Then*

$$\begin{aligned}(-1)^\ell \gamma_\ell(a) = &\sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j(-n)_j}{(-n-m)_j} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a) + \\ &\sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^{m} \frac{(-m)_j}{(-n-m)_j} \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j,k+1) \\ &+ \frac{\ln^{\ell+1} a}{\ell+1}(-1)^{\ell+1} + R_{\ell,m,n,a}, \end{aligned} \tag{4}$$

where $s(j,k)$ are the Stirling numbers of the first kind defined by $(x)_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k$ and the remainder term is

$$R_{\ell,m,n,a} = \frac{(-1)^{m+1}}{(m+n)!} \sum_{k=0}^{\ell} s(m+n+1,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \int_0^1 x^m (1-x)^n \zeta^{(\ell-k,0)}(m+n+1,a+x)dx. \tag{5}$$

The Stirling numbers satisfies also some known properties used in this paper:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} s(n,k) = \frac{(\ln(1+x))^k}{k!}, \quad |x| < 1, \tag{6}$$

$$s(n,1) = (-1)^{n-1}(n-1)!, \tag{7}$$

$$s(n,2) = (-1)^n (n-1)! H_{n-1}, \tag{8}$$

where $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j} = \int_0^1 \frac{1-t^j}{1-t} dt$ is the Harmonic number.

Particular case. $\ell = 0$

$$\begin{aligned}\gamma(a) = &- \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j(-n)_j}{(-n-m)_j} \frac{1}{j} \zeta(j,a) + \sum_{j=1}^{m} \frac{(-m)_j}{(-n-m)_j} \frac{a^{-j}}{j} - \\ &- \ln a + R_{0,m,n,a}, \end{aligned} \tag{9}$$

where

$$R_{0,m,n,a} = (-1)^n \int_0^1 x^m (1-x)^n \zeta(m+n+1, a+x) dx.$$

For $a = 1$, formula (9) becomes

$$\gamma = -\sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j(-n)_j}{(-n-m)_j}\frac{1}{j}\zeta(j) + \sum_{j=1}^{m} \frac{(-m)_j}{(-n-m)_j}\frac{1}{j} + \varepsilon_{n,m}, \tag{10}$$

where $\varepsilon_{n,m} = (-1)^n \int_0^1 x^m(1-x)^n \zeta(m+n+1, 1+x)dx$.

The previous formula has been proved in [5, p. 512].

3. proof of Theorem 2

First, for $p = 1$, we divide the relation (3) by s, we replace σ by $\sigma - 1$ and we compute the ℓ-th derivative at $\sigma = 1$.

$$\frac{A^{(1)}_{(n,m)}(\sigma-1,a)}{\sigma-1} = \sum_{j=0}^{m} \frac{(-m)_j}{(-n-m)_j}\frac{(\sigma)_{j-1}}{j!}\zeta(\sigma-1+j,a+1) - \sum_{j=0}^{n} \frac{(-n)_j(-1)^j}{(-n-m)_j}\frac{(\sigma)_{j-1}}{j!}\zeta(\sigma-1+j,a) \tag{11}$$

$$= (-1)^{n+1}\frac{(\sigma)_{m+n}}{(n+m)!}\int_0^1 x^m(1-x)^n\zeta(\sigma+m+n,a+x)dx. \tag{12}$$

In the relation (11), the term for $j = 0$ is $\frac{1}{\sigma-1}\zeta(\sigma-1,a+1) - \frac{1}{\sigma-1}\zeta(\sigma-1,a) = \frac{-1}{\sigma-1}a^{-\sigma+1}$.
For $j = 1$, it is $\frac{-m}{-n-m}\zeta(\sigma,a+1) - \frac{n}{-n-m}\zeta(\sigma,a) = \zeta(\sigma,a) - \frac{m}{n+m}a^{-\sigma}$.
Thus

$$\left(\frac{d}{d\sigma}\right)^{\ell} \frac{A^{(1)}_{(n,m)}(\sigma-1,a)}{\sigma-1}\Bigg|_{\sigma=1} = \left(\frac{d}{d\sigma}\right)^{\ell}\left(\zeta(\sigma,a) - \frac{a^{-\sigma+1}}{\sigma-1} - \frac{m}{n+m}a^{-\sigma}\right)\Bigg|_{\sigma=1}$$
$$+ \sum_{j=2}^{m} \frac{(-m)_j}{(-n-m)_j}\frac{1}{j!}\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}\zeta(\sigma-1+j,a+1)\Bigg|_{\sigma=1}$$
$$- \sum_{j=2}^{n} \frac{(-n)_j}{(-n-m)_j}\frac{(-1)^j}{j!}\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}\zeta(\sigma-1+j,a)\Bigg|_{\sigma=1}.$$

Using the relation $\zeta(\sigma,a+1) = \zeta(\sigma,a) - a^{-\sigma}$, we find

$$\left(\frac{d}{d\sigma}\right)^{\ell} \frac{A^{(1)}_{(n,m)}(\sigma-1,a)}{\sigma-1}\Bigg|_{\sigma=1} = \left(\frac{d}{d\sigma}\right)^{\ell}\left(\zeta(\sigma,a) - \frac{a^{-\sigma+1}}{\sigma-1}\right)\Bigg|_{\sigma=1}$$
$$+ \sum_{j=2}^{m} \frac{(-m)_j - (-n)_j(-1)^j}{(-n-m)_j}\frac{1}{j!}\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}\zeta(\sigma-1+j,a)\Bigg|_{\sigma=1}$$
$$- \sum_{j=1}^{m} \frac{(-m)_j}{(-n-m)_j}\frac{1}{j!}\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}a^{-\sigma-j+1}\Bigg|_{\sigma=1}.$$

Now, we use the formula [4, Lemma 1]

$$\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}\Bigg|_{\sigma=1}=s(j,\ell+1)\ell!(-1)^{j-1+\ell},j\geq 1.$$

Then, with the product rule, we obtain

$$\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}\zeta(\sigma-1+j,a)\Bigg|_{\sigma=1}=\sum_{k=0}^{\ell}s(j,k+1)\frac{\ell!}{(\ell-k)!}(-1)^{j-1+k}\zeta^{(\ell-k)}(j,a),j\geq 2,$$

and

$$\left(\frac{d}{d\sigma}\right)^{\ell}(\sigma)_{j-1}a^{-\sigma-j+1}\Bigg|_{\sigma=1}=\sum_{k=0}^{\ell}\frac{\ell!}{(\ell-k)!}(\ln a)^{\ell-k}(-1)^{j+\ell+1}a^{-j}s(j,k+1),j\geq 1,$$

$$\begin{aligned}\left(\frac{d}{d\sigma}\right)^{\ell}\left(\zeta(\sigma,a)-\frac{a^{-\sigma+1}}{\sigma-1}\right)\Bigg|_{\sigma=1}&=\left(\frac{d}{d\sigma}\right)^{\ell}\left(\zeta(\sigma,a)-\frac{1}{\sigma-1}+\frac{1-a^{-\sigma+1}}{\sigma-1}\right)\Bigg|_{\sigma=1}\\&=(-1)^{\ell}\gamma_{\ell}(a)+(-1)^{\ell}\frac{\ln^{\ell+1}a}{\ell+1}.\end{aligned}$$

It arises

$$\begin{aligned}(-1)^{\ell}\gamma_{\ell}(a)=&-\frac{\ln^{\ell+1}a}{\ell+1}(-1)^{\ell}+\sum_{k=0}^{\ell}\frac{\ell!}{(\ell-k)!}\ln^{\ell-k}a\sum_{j=1}^{m}\frac{(-m)_j}{(-n-m)_j}\frac{(-1)^{j+\ell+1}a^{-j}}{j!}s(j,k+1)\\&+\sum_{j=2}^{\max(m,n)}\frac{(-m)_j-(-1)^j(-n)_j}{(-n-m)_j}\frac{(-1)^{j+1}}{j!}\sum_{k=0}^{\ell}s(j,k+1)\frac{\ell!}{(\ell-k)!}(-1)^k\zeta^{(\ell-k,0)}(j,a)+R_{\ell,m,n,a}\end{aligned}$$

where

$$R_{\ell,m,n,a}=\frac{(-1)^{m+1}}{(m+n)!}\sum_{k=0}^{\ell}s(m+n+1,k+1)\frac{\ell!}{(\ell-k)!}(-1)^k\int_0^1x^m(1-x)^n\zeta^{(\ell-k,0)}(m+n+1,a+x). \tag{13}$$

The derivation of $\frac{A^{(1)}_{(n,m)}(\sigma-1,a)}{\sigma-1}$ with respect to σ is justified by Lemma 3 since the function $\phi(x)$ is integrable.

In the following, we slightly modify Lemma 1 for $p=1$.

Lemma 2. *Suppose that $\Re(a)>0$ and $|a|\geq 1$. Then $\forall k\in[0,\ell],\forall m,n$ such that $m+n\geq l+1$,*

$$\left|\int_0^1x^m(1-x)^n\zeta(m+n+1+k-\ell,a+x)dx\right|\leq K\frac{1}{|a+1|^m}\frac{1}{|a|^n},$$

where K is some constant independent of m and n.

Proof.

$$\left|\int_0^1 x^m(1-x)^n\zeta(m+n+1+k-\ell,a+x)dx\right| \leq \int_0^1 x^m(1-x)^n\left|\zeta(m+n+1+k-\ell,a+x)\right|dx$$

$$\leq \int_0^1 \left|\frac{x}{a+x}\right|^m \left|\frac{1-x}{a+x}\right|^n |a+x|^{l-k-1}\left|\zeta(m+n+1+k-\ell,a+x)\right| |a+x|^{m+n+1+k-l}\,dx.$$

In [7], we proved that for $\Re(z)\geq 1, |\zeta(z,a+x)(a+x)^z|$ is bounded by some constant C independent of z and that $\max_{0\leq x\leq 1}\left|\frac{x}{a+x}\right| = \frac{1}{|a+1|}$ and $\max_{0\leq x\leq 1}\left|\frac{1-x}{a+x}\right| = \frac{1}{|a|}$. This proves the Lemma, where $K = C\int_0^1 |a+x|^{l-k-1}\,dx$.

□

In order to prove the convergence to 0 of the remainder term $R_{l,m,n,a}$ of our principal formula (4), we need a preliminary technical result.

Lemma 3. *Suppose that $r\in\mathbb{N}$, $a\in\mathbb{C}, \Re(a)>0$ and $|a|\geq 1$, $j>r+1$.*
Then $\forall x\geq 0, |a+x|\geq 1$, and

$$\left|\zeta^{(r,0)}(j,a+x)\right| \leq \sum_{k\geq 0}\frac{((\ln|k+a+x|)^2+\pi^2/4)^{r/2}}{|k+a+x|^j} =: \phi(x) \tag{14}$$

$$\phi(x)\leq 2^r\zeta(j-r,\Re(a)+x) \tag{15}$$

Proof.

$$\begin{aligned}
\left|\zeta^{(r,0)}(j,a+x)\right| &= \left|\sum_{k\geq 0}(-1)^r\frac{(\ln(k+a+x)^r}{(k+a+x)^j}\right| \leq \sum_{k\geq 0}\frac{|s(k+a+x)|^r}{|k+a+x|^j}\\
&= \sum_{k\geq 0}\frac{|\ln|k+a+x|+iArg(k+a+x)|^r}{|k+a+x|^j}\\
&= \sum_{k\geq 0}\frac{(\ln|k+a+x|)^2+(Arg(k+a+x))^2)^{r/2}}{|k+a+x|^j}\\
&\leq \sum_{k\geq 0}\frac{(\ln|k+a+x|)^2+(\pi/2)^2)^{r/2}}{|k+a+x|^j} \leq \sum_{k\geq 0}\frac{(|k+a+x|^2+3|k+a+x|^2)^{r/2}}{|k+a+x|^j}\\
&\leq 2^r\sum_{k\geq 0}\frac{|k+a+x|^r}{|k+a+x|^j} = 2^r\sum_{k\geq 0}\frac{1}{|k+a+x|^{j-r}}\\
&\leq 2^r\sum_{k\geq 0}\frac{1}{(k+\Re(a)+x)^{j-r}} = 2^r\zeta(j-r,\Re(a)+x)
\end{aligned}$$

□

Theorem 3. *If $\Re(a)\geq 1$ then*

$$\forall n\in\mathbb{N}, \lim_{m\to\infty} R_{\ell,m,n,a} = 0,$$

$$\forall m \in \mathbb{N}, \lim_{n\to\infty} R_{\ell,m,n,a} = 0.$$

Proof. From (13), we can write

$$|R_{\ell,m,n,a}| \le \frac{1}{(m+n)!}\sum_{k=0}^{\ell} |s(m+n+1,k+1)| \frac{\ell!}{(\ell-k)!}\int_0^1 x^m(1-x)^n \left|\zeta^{(\ell-k,0)}(m+n+1,a+x)\right| dx$$

□

Adell in [1], proved some explicit upper bounds for the Stirling numbers of the first kind. We will use the following formula:

$$for\ l = 1, \cdots, j-1, \quad |s(j+1,\ell+1)| \le \frac{j!}{\ell!}(\ln j)^\ell \left(1+\frac{\ell}{\ln j}\right).$$

$$|R_{\ell,m,n,a}| \le \sum_{k=0}^{\ell}(\ln(m+n))^k\left(1+\frac{k}{\ln(m+n)}\right)\frac{1}{(\ell-k)!}\int_0^1 x^m(1-x)^n\left|\zeta^{(\ell-k,0)}(m+n+1,a+x)\right| dx$$

$$\le \left(1+\frac{\ell}{\ln(m+n)}\right)(\ln(m+n))^\ell\sum_{k=0}^{\ell}2^{\ell-k}\int_0^1 x^m(1-x)^n\zeta(m+n+1-\ell+k,\Re(a)+x)$$

(Lemma 3)

$$\le C\left(1+\frac{\ell}{\ln(m+n)}\right)(\ln(m+n))^\ell 2^{\ell+1}\frac{1}{(\Re(a)+1)^m}\frac{1}{\Re(a)^n}. \quad (16)$$

The second inequality is valid since m and/or n tends to infinity and thus the parameter $m+n+1+k-\ell$ is greater that 1.

Thus, if $\Re(a) > 1$, the limit of the remainder $R_{\ell,mn,a}$ is 0 when m or n tend to infinity and we have

$$\lim_{m\to\infty}|R_{\ell,m,n,a}|^{1/m} \le \frac{1}{\Re(a)+1},$$

$$\lim_{n\to\infty}|R_{\ell,m,n,a}|^{1/n} \le \frac{1}{\Re(a)}.$$

If $\Re(a) = 1$, then $\lim_{m\to\infty}|R_{\ell,m,n,a}|^{1/m} \le 1/2$.

Now, if $\Re(a) = 1$ and n tends to infinity, we have to bound the integral term of right hand side of (16) as following.

$$\int_0^1 x^m(1-x)^n\zeta(m+n+1+k-\ell,1+x)dx \le \int_0^1(1-x)^n\zeta(m+n+1+k-\ell,1+x)dx$$

$$\le \int_0^1\frac{(1-x)^n}{(1+x)^{m+n+1+k-\ell}}(1+x)^{m+n+1+k-\ell}\zeta(m+n+1+k-l,1+x)dx$$

$$\leq \int_0^1 \frac{(1-x)^n}{(1+x)^n}(1+x)^{m+n+1+k-\ell}\zeta(m+n+1+k-\ell,1+x)dx$$

$$\leq C\int_0^1 \frac{(1-x)^n}{(1+x)^n}dx \leq \frac{C}{2n}.$$

(see Lemma 2).

We can conclude that, when $\Re(a)=1$ the remainder term tends to 0 when n tends to infinity.

Remark

To accelerate the computation, we can use the following relation (for p integer):

$$\gamma_\ell(a) = \gamma_\ell(a+p) + \sum_{k=0}^{p-1}\frac{\ln^\ell(a+k)}{a+k}.$$

It leads to

$$(-1)^\ell\gamma_\ell(a) = (-1)^\ell\gamma_\ell(a+p) + (-1)^\ell\sum_{k=0}^{p-1}\frac{\ln^\ell(a+k)}{a+k}$$

$$= (-1)^\ell\sum_{k=0}^{p-1}\frac{\ln^\ell(a+k)}{a+k}+$$

$$\sum_{j=2}^{\max(m,n)}\frac{(-m)_j-(-1)^j(-n)_j}{(-n-m)_j}\frac{(-1)^{j+1}}{j!}\sum_{k=0}^{\ell}S(j,k+1)\frac{\ell!}{(\ell-k)!}(-1)^k\zeta^{(\ell-k,0)}(j,a+p)+$$

$$\sum_{k=0}^{\ell}\frac{\ell!}{(\ell-k)!}\ln^{\ell-k}(a+p)\sum_{j=1}^{m}\frac{(-m)_j}{(-n-m)_j}\frac{(-1)^{j+\ell+1}(a+p)^{-j}}{j!}S(j,k+1)+R_{\ell,m,n,a+p},$$

where the remainder terms $R_{\ell,m,n,a+p}$ converges to 0 as $\dfrac{1}{\Re(a+p+1)^m\Re(a+p)^n}$

4. Particular cases

In this section, we consider all the possible values for the four parameters m, n, a.

4.1. $m \in \mathbb{N}, n = 0, \Re(a) \geq 1$. After simplification, we get

$$
\begin{aligned}
(-1)^{\ell}\gamma_{\ell}(a) = & \sum_{j=2}^{m} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a) \\
& + \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^{m} \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j,k+1) \\
& + \frac{\ln^{\ell+1} a}{\ell+1}(-1)^{\ell+1} + R_{\ell,m,0,a}.
\end{aligned} \tag{17}
$$

If m tends to infinity, it arises (see Theorem 3)

$$
\begin{aligned}
(-1)^{\ell}\gamma_{\ell}(a) = & \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a) \\
& + \sum_{k=0}^{\ell} \frac{\ell!}{(\ell-k)!} \ln^{\ell-k} a \sum_{j=1}^{\infty} \frac{(-1)^{j+\ell+1} a^{-j}}{j!} s(j,k+1) \\
& + \frac{\ln^{\ell+1} a}{\ell+1}(-1)^{\ell+1}.
\end{aligned} \tag{18}
$$

After simplification (using relation (6)) leads to

$$
(-1)^{\ell}\gamma_{\ell}(a) = \frac{\ln^{\ell+1}(a-1)}{\ell+1}(-1)^{\ell+1} + \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{\ell} s(j,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a).
$$

4.2. $n \in \mathbb{N}, m = 0, \Re(a) \geq 1$. After simplification, we get

$$
(-1)^{\ell}\gamma_{\ell}(a) = \sum_{j=2}^{n} \frac{1}{j!} \sum_{k=0}^{\ell} s(j,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a) + \frac{\ln^{\ell+1} a}{\ell+1}(-1)^{\ell+1} + R_{\ell,0,n,a}.
$$

If n tends to infinity, it arises (see Theorem 3)

$$
(-1)^{\ell}\gamma_{\ell}(a) = \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{k=0}^{\ell} s(j,k+1) \frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a) + \frac{\ln^{\ell+1} a}{\ell+1}(-1)^{\ell+1}.
$$

5. Case $\Re(a) > 0, |a| \geq 1, m = \lambda n, m, n \to \infty$

Now, we consider the case $m = \lambda n$, when n tends to infinity.

Of course, the general expression of Theorem 2 is true and we can derive the following series

$$\begin{aligned}(-1)^\ell \gamma_\ell(a) &= -\frac{\ln^{\ell+1} a}{\ell+1}(-1)^\ell + \frac{\lambda}{\lambda+1}\frac{\ln^\ell a}{a}(-1)^\ell \\ &- \sum_{j=2}^{\infty} \frac{(\lambda)^j - (-1)^j}{(\lambda+1)^j}\frac{(-1)^{j+1}}{j!}\sum_{k=0}^{\ell} s(j,k+1)\frac{\ell!}{(\ell-k)!}(-1)^k \zeta^{(\ell-k,0)}(j,a) \\ &+ \sum_{k=0}^{\ell}(-1)^{\ell+1}\frac{\ell!}{(\ell-k)!}\ln^{\ell-k}(a)\sum_{j=2}^{\infty}\left(\frac{-\lambda/a}{\lambda+1}\right)^j \frac{1}{j!}s(j,k+1).\end{aligned}$$

Using the relation (6) and for $\left|\frac{\lambda/a}{\lambda+1}\right| < 1$, it can be simplified as

$$(-1)^\ell \gamma_\ell(a) = (-1)^{\ell+1}\frac{\ln^{\ell+1}\left(a - \frac{\lambda}{\lambda+1}\right)}{\ell+1} - \sum_{k=0}^{\ell}\frac{\ell!(-1)^k}{(\ell-k)!}\left[\sum_{j=2}^{\infty}\frac{(\lambda)^j - (-1)^j}{(\lambda+1)^j}\frac{(-1)^{j+1}}{j!}s(j,k+1)\zeta^{(\ell-k,0)}(j,a)\right]. \tag{19}$$

One may consider (19) as an identity between two holomorphic functions in the variable λ. Since this formula is true for $\lambda \in \mathbb{Q}$ such that $\left|\frac{\lambda}{\lambda+1}\right| < |a|$, it is obviously true for every $\lambda \in \mathbb{C}, \left|\frac{\lambda}{\lambda+1}\right| < |a|$.

Now, suppose that a is real greater than 1 and we choose $\lambda = i$.

Equating real and imaginary parts of (19) we get

$$\begin{aligned}(-1)^\ell \gamma_\ell(a) &= (-1)^{\ell+1}\Re\left(\frac{\ln^{\ell+1}(a - 1/2 - i/2)}{\ell+1}\right) \\ &- \sum_{k=0}^{\ell}\frac{\ell!(-1)^k}{(\ell-k)!}\left[\begin{array}{c}\sum_{q=1}^{\infty}(-1)^q 2^{-2q}\frac{1}{(4q+1)!}s(4q+1,k+1)\zeta^{(\ell-k,0)}(4q+1,a) \\ -\sum_{q=0}^{\infty}(-1)^q 2^{-2q-1}\frac{1}{(4q+3)!}s(4q+3,k+1)\zeta^{(\ell-k,0)}(4q+3,a)\end{array}\right]\end{aligned}$$

and the following identity between Hurwitz-ζ functions:

$$\sum_{k=0}^{\ell}\frac{l!}{(l-k)!}(-1)^k\sum_{j\equiv 2(4)}^{\infty}(-1/4)^{\frac{j-2}{4}}\frac{(-1)^{j-1}}{j!}s(j,k+1)\zeta^{(\ell-k,0)}(j,a) = \Im\left((-1)^{\ell+1}\frac{\ln(a - \frac{i+1}{2})^{\ell+1}}{\ell+1}\right), \tag{20}$$

or more simply

$$\sum_{k=0}^{\ell}\frac{l!}{(l-k)!}(-1)^k\sum_{r=0}^{\infty}\frac{(-1/4)^r}{(4r+2)!}s(4r+2,k+1)\zeta^{(\ell-k,0)}(4r+2,a) = \Im\left((-1)^{\ell}\frac{\ln(a - \frac{i+1}{2})^{\ell+1}}{\ell+1}\right). \tag{21}$$

6. Particular case: $\ell = 1$

In this section, we consider the first Stieltjes constant γ_1. We will show a new formula of this constant in terms of the first derivatives of the ζ-functions.

Proposition.

$$\gamma_1 = \sum_{j=1}^{\infty} \frac{\zeta'(2j+1)}{2j+1}.$$

Proof. We use the formula (19) with $a = 1$, $l = 1$ and $\lambda = 0$.

$$\gamma_1 = \sum_{j=2}^{\infty} \frac{1}{j!}(s(j,1)\zeta'(j) - s(j,2)\zeta(j)) \tag{22}$$

The formula (19) with $a = 2$, $l = 1$ and $\lambda \to \infty$.

$$\gamma_1(2) = \gamma_1 = \sum_{j=2}^{\infty} \frac{1}{j!}(-1)^{j-1}(s(j,1)\zeta'(j,2) - s(j,2)\zeta(j,2)) \tag{23}$$

$$= \sum_{j=2}^{\infty} \frac{1}{j!}(-1)^{j-1}(s(j,1)\zeta'(j) - s(j,2)\zeta(j,2)) \tag{24}$$

Using the expression of the first Stirling numbers (7, 8), the two expression of γ_1 become

$$\gamma_1 = \sum_{j=2}^{\infty} \frac{(-1)^{j-1}}{j}(\zeta'(j) + H_{j-1}\zeta(j)) \tag{25}$$

$$\gamma_1 = \sum_{j=2}^{\infty} \frac{1}{j}(\zeta'(j) + H_{j-1}\zeta(j,2)) \tag{26}$$

In the sequel, we will show that

$$\sum_{j=2}^{\infty} \frac{H_{j-1}}{j}\left(\zeta(j,2) + (-1)^{j-1}\zeta(j)\right) = 0$$

and the proposition will be proved.

$$\sum_{j=2}^{\infty} \frac{H_{j-1}}{j}\left(\zeta(j,2) + (-1)^{j-1}\zeta(j)\right) = \int_0^1 \sum_{j=2}^{\infty} \frac{1}{j}\left(\zeta(j,2) + (-1)^{j-1}\zeta(j)\right)\frac{1-t^{j-1}}{1-t}dt.$$

The permutation is valid since the convergence of the series is uniform on $[0,1]$.
Using the formula (https://dlmf.nist.gov/5.7.E3)

$$\ln\Gamma(1+z) = -\ln(1+z) + z(1-\gamma) + \sum_{k=2}^{\infty}(-1)^k\zeta(k,2)\frac{z^k}{k}, \quad |z| < 2,$$

it is easy to prove that

$$\sum_{j\geq 2}\frac{1}{j}(\zeta(j,2)+(-1)^{j-1}\zeta(j))\frac{1-t^{j-1}}{1-t}=\sum_{j\geq 2}\frac{1}{j}\zeta(j,2)\frac{1-t^{j-1}}{1-t}+\sum_{j\geq 2}\frac{(-1)^{j-1}}{j}\zeta(j)\frac{1-t^{j-1}}{1-t}$$
$$=-\frac{\ln\Gamma(2-t)}{t(1-t)}+\frac{\ln\Gamma(1+t)}{t(1-t)}=:\phi(t).$$

This function ϕ is symetric with respect to the abscissa $t=1/2$, so its integral between 0 and 1 is zero and the proposition is proved. □

References

[1] Adell, J. A. Explicit upper bounds for the Stirling numbers of the first kind. *J. Combin. Theory Ser. A 192* (2022), Paper No. 105669, 17.

[2] Adell, J. A., and Lekuona, A. Fast computation of the Stieltjes constants. *Math. Comp. 86*, 307 (2017), 2479–2492.

[3] Berndt, B. C. On the Hurwitz zeta-function. *Rocky Mountain J. Math. 2*, 1 (1972), 151–157.

[4] Coffey, M. W. Series representations for the Stieltjes constants. *Rocky Mountain J. Math. 44*, 2 (2014), 443–477.

[5] Elsner, C., and Prévost, M. Expansion of Euler's constant in terms of zeta numbers. *J. Math. Anal. Appl. 398*, 2 (2013), 508–526.

[6] Johansson, F., and Blagouchine, I. V. Computing Stieltjes constants using complex integration. *Math. Comp. 88*, 318 (2019), 1829–1850.

[7] Prévost, M. Recurrence for values of the zeta function. *Appl. Numer. Math. 60*, 12 (2010), 1382–1394.

[8] Prévost, M., and Rivoal, T. New convergent sequences of approximations to Stieltjes' constants. *J. Math. Anal. Appl. 524*, 2 (2023), Paper No. 127091, 20.

[9] Wilton, J. R. A note on the coefficients in the expansion of $\zeta(s,x)$ in powers of $s-1$. *Quart. J. Pure Appl. Math. 50* (1927), 329–332.

Email address: `marc.prevost (a) univ-littoral.fr`

M. Prévost,
Univ. Littoral Côte d'Opale, UR 2597-LMPA-Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100, Calais, France
CNRS, FR2037, Calais, France